Bayesian sediment transport model for unisize bed load

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[1] Fluvial sediment transport studies have long underscored the difficulty in reliably estimating transport model parameters, collecting accurate observations, and making predictions because of measurement error, natural variability, and conceptual model uncertainty. Thus, there is a need to identify modeling frameworks that accommodate these realities while incorporating functional relationships, providing probability-based predictions, and accommodating for conceptual model discrimination. Bayesian statistical approaches have been widely used in a number of disciplines to accomplish just this, yet applications in sediment transport are few. In this paper we propose and demonstrate a Bayesian statistical approach to a simple sediment transport problem as a means to overcome some of these challenges. This approach provides a means to rigorously estimate model parameter distributions, such as critical shear, given observations of sediment transport; provides probabilistically based predictions that are robust and easily interpretable; facilitates conceptual model discrimination; and incorporates expert judgment into model inference and predictions. We demonstrate a simple unisize sediment transport model and test it against simulated observations for which the "true" model parameters are known. Experimental flume observations were also used to assess the proposed model's robustness. Results indicate that such a modeling approach is valid and presents an opportunity for more complex models to be built in the Bayesian framework.

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1. Introduction

[2] Among the challenges communicated in the literature regarding sediment transport, dealing with uncertainty is a recurring theme. The scope of this uncertainty encompasses many aspects, including our ability to make accurate measurements [*Bunte and Abt*, 2001], the effect of natural variation or stochasticity on the process [*Singh*, 2009], and our ability to distill the true underlying relationship that governs the complex phenomenon into a conceptual model [*Gomez and Church*, 1989]. Because of these uncertainties, there is a need to identify a framework in which sediment transport models can accommodate notions of uncertainty, variability, and error.

[3] In recent years, Bayesian statistical models have been employed to address these challenges in a wide variety of disciplines. Researchers in ecology, hydrology, and atmospheric and environmental science have increasingly used the Bayesian framework to model complex phenomena, for example, distributed rainfall-runoff models, species invasion dynamics, and uncertainty estimation in climate models [*Cressie et al.*, 2009; *Hooten and Wikle*, 2007; *Renard et al.*, 2010; *Smith et al.*, 2009; *Wikle and Hooten*, 2010; *Vrugt et al.*, 2008]. Despite the mature models described in the hydrology and other literature, Bayesian methods have not been widely applied to sediment transport problems. This modeling framework was first mentioned by *Griffiths* [1982] in the context of geomorphology and has since been employed by only a few, including *Fox and Papanicolaou* [2008], *Kanso et al.* [2005], and *Wu and Chen* [2009]. *Wu and Chen* [2009] also recognized the limited use of Bayesian methods for sediment transport problems and supposed that that this may be due to a lack of demonstration to the sediment transport community.

1.1. Determinism and Statistics

[4] For proper context, it is helpful to draw a distinction between purely deterministic and traditional and Bayesian statistical methods. Purely deterministic methods (in the sense of Laplacian determinism [*Laplace*, 1825]) assume that any functional or physically based relationship used in an analysis is completely representative of the system and that any associated parameter values are fixed and known. Regarding our ability to discern this true underlying process, G. Box stated that "all models are wrong, but some are useful." [*Box and Draper*, 1987, p. 424]. Modern statistical methods (traditional and Bayesian alike) recognize this and provide means to make predictions and inference using noisy observations and imprecise conceptual models.

[5] Bayesian models, like traditional methods such as maximum likelihood and regression techniques, are robust statistical methods that provide a means for parameter inference and prediction. When employed in very simple

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scenarios, such as the current sediment transport model, both approaches are likely to yield convergent solutions (M. L. Schmelter and D. K. Stevens, Traditional and Bayesian statistical models in fluvial sediment transport, submitted to *Journal of Hydraulic Engineering*, 2011). As deterministic models become more involved (e.g., the number of latent parameters increases and/or nested functions are employed), Bayesian models constructed hierarchically provide elegant and robust solutions, as evidenced by the number of applications in the hydrology literature in recent years. In this paper we provide a simple example of how the Bayesian framework can be employed in sediment transport and discuss several of the motivating factors for why they should be employed. Our thesis is that the simplicity of the proposed model will provide a demonstrated foundation upon which future sediment transport models that incorporate increased complexity may be built.

1.2. Challenges in Sediment Transport

[6] Sediment transport is faced with many of the same challenges faced by the hydrology community, which has enjoyed wide use of Bayesian methods. Below, we outline a few of these challenges as they relate to the field of sediment transport that can be addressed through Bayesian methods.

[7] Like other disciplines, sediment transport modeling involves the estimation of model parameters. For instance, many transport models require the specification of the critical Shields number, a quantity that cannot be directly measured [*Wilcock*, 1988]. Deterministic approaches to modeling transport entails the point estimation of model parameters. *Mueller et al.* [2005], however, noted that uncertainties in specifying a single critical shear can lead to large errors since the process is nonlinear. This estimate can be made using statistical methods, but it is not uncommon to adjust it by eye [*Erwin et al.*, 2011; *Gaeuman et al.*, 2009; *Wilcock et al.*, 2003; *Mueller et al.*, 2005; *Wilcock et al.*, 2009; *Wilcock*, 1988] because of some nonlinear regression techniques giving too much weight to observations far from critical (see *Wilcock* [1988] for a discussion on this). Additionally, *Buffington and Montgomery* [1997] compiled data from eight decades of incipient motion studies and created lists of reported critical shear values. They noted that the early researchers in the field [*Shields*, 1936; *Einstein*, 1950; *Grass*, 1970; *Gessler*, 1971; *Paintal*, 1971] acknowledged that the threshold at which sediments move is "inherently a statistical problem" and asserted that a frequency distribution of critical shear values for any given grain size is more appropriate than any single value (the physical basis for which is described by *Johnston et al.* [1998] and *Buffington et al.* [1992]). *Lavelle and Mofjeld* [1987] questioned the idea of whether or not a single threshold exists at all and stated that stochastic aspects of transport lead us to reason that the critical shear concept will not be sharply defined and that it is most appropriately described using statistical distributions. Critical shear will not be sharply defined in applied settings because of constantly changing hydraulic conditions, including changing bed topography, turbulence, varying supplies of bed material from upstream processes, and irreducible noise brought about by stochasticity [*Bunte and Abt*, 2005; *Diplas et al.*, 2008; *Gomez and Phillips*, 1999; *Grass*, 1970; *Kirchner et al.*, 1990; *Knighton*, 1998; *Hicks and Gomez*, 2005; *McLean et al.*, 1999].

[8] Making observations of transport events is also problematic. In addition to the expense and difficulty in making sediment transport observations, measurements of sediment transport are prone to substantial uncertainty [*Gaeuman et al.*, 2009; *Wilcock*, 2001]. Variation is inevitably caused by sampling error as well as natural stochasticity. Variability, distinct from uncertainty, is an intrinsically important characteristic of natural systems and should be considered when modeling these systems [*Clement and Piegay*, 2005].

[9] Recent work in hydrology underscores the difficulty of characterizing structural uncertainty [*Renard et al.*, 2010], which is uncertainty due to model lack of fit, and the same holds for sediment transport. Numerous bed load sediment transport relations have been suggested [see, e.g., *Garcia*, 2008; *Gomez and Church*, 1989]; thus, there is a need to select one relationship over another, and therefore, some degree of uncertainty and judgment is associated with this selection.

[10] Last, sediment transport is known to be variable even under steady state conditions [*Knighton*, 1998; *Hicks and Gomez*, 2005; *Turowski*, 2010], and predictions that are based on a purely deterministic framework do not necessarily account for this variability. In this case, predictive distributions with quantifiable probabilities are more appropriate than a single line fitted through a spread of observations.

2. Purpose and Objectives

[11] The goal of this research is to incorporate and address many of the elements introduced in section 1 and develop them into a fluvial sediment transport model as a demonstration of the Bayesian statistical method applied to sediment transport problems, which can then be used as a foundation on which to build more complex models. In particular, our objectives are described below.

[12] 1. The first objective is to demonstrate the development and implementation of a Bayesian sediment transport model that when given transport observations, makes it possible (1) to estimate a credible interval (the Bayesian analog to confidence intervals, which is interpreted as the probability the realized parameter value is found in the specified interval) for the critical shear parameter τ_c, (2) to estimate a credible interval for the variance parameter σ^2, a measure of model misspecification, measurement error, and random variation, (3) to provide sediment transport predictions delineated in credible intervals, (4) to compare different process models for fit via quantitative metrics, and (5) to easily extend or generalize to accommodate more complex descriptions.

[13] 2. Our second objective is to perform simulation studies to evaluate the proposed framework, specifically, (1) simulate synthetic data according to established transport relationships with multiplicative noise according to σ^2, (2) validate the model (verify that the model can recover the parameters that were specified when the synthetic data were generated), and (3) explore the effect of various specifications for prior information on model inference.

[14] 3. The last objective is to evaluate the model using observed transport data from flume studies with unisize sediment: (1) estimate model parameters τ_c and σ^2, (2) evaluate different process models, and (3) provide a sediment rating curve in terms of credible intervals.

3. Formulation of the Bayesian Model

[15] Though Bayesian methods have appeared in the broader literature, we take a pedagogical approach here with a specific focus on sediment transport. To clearly explain and illustrate how such a model is developed and implemented, we have divided section 3 into the following subtopics: (1) governing sediment transport relations, which describe the mathematical formulations we used to model sediment transport from a physical and deterministic reference, (2) the specification of the Bayesian model, with considerations to basic Bayesian modeling concepts as well as likelihood and prior distribution selection, (3) data simulation, which discloses the process for creating synthetic data from the governing sediment transport relations and known parameters, (4) computational methods, which provide a means to integrate the multidimensional relationships that naturally result from a Bayesian model, and (5) last, a means for model evaluation whereby the predictive capabilities of the sediment transport model can be assessed. These subtopics will be described in what follows.

3.1. Governing Sediment Transport Relations

[16] We emphasize at this point that the present research is not about modeling the fine-resolution physics of sediment entrainment; rather, it is about larger-scale bulk transport. While there are several options for the physical description of sediment transport, this paper uses an excess shear model because of its wide recognition in the literature.

[17] The basic principle of an excess shear model is that the amount of sediment transport is related, nonlinearly, to the difference between the force induced on the grains due to the overlying movement of water and the amount of force required to move sediment of an arbitrary size [e.g., *Wilcock et al.*, 2009]. Mathematically, this relationship is

$$q^* = a(\tau^* - \tau_c^*)^b, \tag{1}$$

where q^* is the Einstein transport parameter, τ^* is the Shields number, and τ_c^* is the critical Shields number. The parameters a and b are empirical coefficients and can take on a range of values, reflecting the numerous different proposed transport relations. While the individual components of (1) are entirely nondimensional, these parameters can be expressed in dimensional terms:

$$q^* = \frac{q_s}{\sqrt{(s-1)gD^3}}, \tag{2}$$

where q_s is unit sediment discharge in m^2 s^{-1}, s is specific gravity of the sediments, g is acceleration due to gravity in m s^{-2}, and D is particle diameter in m. The Shields number τ^* (as described above) is represented by

$$\tau^* = \frac{\tau}{(s-1)\rho g D}, \tag{3}$$

where τ is the grain shear stress in Pa, and ρ is the fluid density in kg m^{-3}. The critical Shields number quantifies how much shear is required to move an arbitrary particle and is defined as

$$\tau_c^* = \frac{\tau_c}{(s-1)\rho g D}, \tag{4}$$

where τ_c is the critical shear stress in Pa. One method to estimate shear stress experienced by the grains τ is presented by *Wilcock* [2001]:

$$\tau = 0.052\rho(gSD_{65})^{0.25}u^{1.5}, \tag{5}$$

where ρ and g are in SI units, as noted above, u is the mean flow (depth-averaged) velocity in m s^{-1}, S is the surface slope, and D_{65} is the 65th percentile grain size in m. Since the model proposed here is for a bed of unisize grains, D_{65} simply reduces to D. With the components of (1) defined, a fully dimensional relationship can be derived in which q_s can be isolated:

$$q_s = a\sqrt{(s-1)gD^3}\left(\frac{0.052(gSD)^{0.25}u^{1.5}}{(s-1)gD} - \frac{\tau_c}{(s-1)\rho g D}\right)^b. \tag{6}$$

There exist several variations on the excess shear transport model presented in (1) that were explored in this research. While incorporating any number of them would be straightforward, we limit the number of models discussed in this paper to two for pedagogical reasons. These process models are given in (7) and (8) and are from *Meyer-Peter and Müller* [1948] and *Wong and Parker* [2006], respectively:

$$q^* = 8(\tau^* - \tau_c^*)^{3/2}, \tag{7}$$

$$q^* = 4.93(\tau^* - \tau_c^*)^{1.6}. \tag{8}$$

3.2. Specification of Bayesian Transport Model

3.2.1. Bayesian Model Fundamentals

[18] Bayesian modeling is derived from Bayes' theorem, which in a statistical context can be written as

$$[\theta|z] = \frac{[z|\theta][\theta]}{\int [z|\theta][\theta]\mathrm{d}\theta}, \tag{9}$$

where the square brackets are a conventional Bayesian notation denoting a probability distribution and the denominator is an unknown normalizing constant. The model in (9) is conventionally expressed as a proportionality, $[\theta|z] \propto [z|\theta][\theta]$, since the computational methods employed in this paper are able to resolve the underlying posterior distribution without knowing the normalizing constant.

[19] Were we to observe or collect data on the process z, the model in (9) tells us that it would be controlled by some underlying parameter θ. The posterior distribution $[\theta|z]$ describes the distribution of values that θ can assume, given the observations of z. It is the posterior distribution that allows us to make inference on the model parameters and thereby establish credible intervals of values for θ. The right-hand side of (9) is composed of two parts: the likelihood and the prior. The likelihood $[z|\theta]$ (i.e., the data model) describes the distribution of the observations, given θ. The likelihood should describe the structure of the process z, given the fact that it is controlled by a parameter θ. Last, the prior distribution $[\theta]$ encompasses what is known about θ before considering the new observations: it can be the summary of all previous research in the literature on θ, or it can simply limit possible ranges of values for θ.

[20] This basic example can be extended to include a vector or parameters $\boldsymbol{\theta}$, such that

$$[\boldsymbol{\theta}|z] \propto [z|\boldsymbol{\theta}][\boldsymbol{\theta}], \quad (10)$$

provided that either an appropriate multivariate prior or individual prior distributions are specified for each element in $\boldsymbol{\theta}$.

3.2.2. Bayesian Transport Model

[21] A Bayesian model for sediment transport can be written as

$$\underbrace{[\tau_c, \sigma^2|\log(\mathbf{q_{s,o}})]}_{\text{Posterior}} \propto \underbrace{\left(\prod_{i=1}^{n}[\log(q_{s,o,i})|\tau_c, \sigma^2]\right)}_{\text{Likelihood}} \underbrace{[\tau_c][\sigma^2]}_{\text{Priors}}, \quad (11)$$

where $\mathbf{q_{s,o}} = (q_{s,o,i}, \ldots, q_{s,o,n})'$. The model specified in (11) makes inference on multiple parameters, τ_c and σ^2, given a set of observations $\mathbf{q_{s,o}}$ of sediment transport. The product of the i likelihoods represents the joint distribution of sediment transport observations. Also, the construction of the model above assumes independence between τ_c and σ^2, thereby allowing us to take the product of the two distributions as their joint prior density (i.e., $[\tau_c, \sigma^2] = [\tau_c][\sigma^2]$). If the parameters cannot be assumed to be independent, the priors may be specified by $[\tau_c, \sigma^2] = [\tau_c|\sigma^2][\sigma^2]$ or some other conditional decomposition where the final prior takes the form of a Jeffreys [*Jeffreys*, 1946] or some other default prior.

[22] While it is theoretically possible to analytically determine the form of the posterior distributions of Bayesian models, this determination is impractical in all but the most simple model formulations (see *Griffiths* [1982] for geomorphic examples of analytically tractable models). In the case of this model, an analytical derivation of the joint posterior distribution was not attempted because of inherent nonlinearities, and the posterior distribution was determined using the sampling methods explained in section 4.

3.2.3. Likelihood

[23] Irrespective of whether one chooses to derive the posterior analytically or not, the likelihood and prior distributions must be specified a priori. The likelihood in this model is specified as a function of the governing equations, including the parameter τ_c, and additive noise (controlled by σ^2) in log space. Specifically, the likelihood is denoted as

$$\log(q_{s,o,i})|\tau_c, \sigma^2 \sim N\big(\log(q_{s,i}), \sigma^2\big), \quad (12)$$

where each observation $\log(q_{s,o,i})$ has a mean value defined by the log() of (6) and variance σ^2. The selection of a normal likelihood is ideal from a computational standpoint because it allows the use of an efficient Gibbs sampler explained in section 4.2. The log() in the likelihood is a modeling assumption in which the error structure is assumed to be additive in log space and is not related to the log likelihood commonly encountered in maximum likelihood techniques.

[24] The model in (12) implies that the sediment transport process generally follows the governing relationship specified in (6), but with some variability. In this case, the variance combines natural variation of the process, measurement error, and model misspecification into one term but makes it possible to separate out the variability in sediment transport due to τ_c being a random variable as opposed to a fixed value. Further separation of variance can be achieved through hierarchical models [*Cressie et al.*, 2009].

3.2.4. Priors

[25] The prior distributions for τ_c and σ^2 must accommodate the physical realities of the parameters they represent. For instance, σ^2 must have positive real support (negative values of variance are not reasonable); thus, an inverse gamma (IG) distribution would be an appropriate selection as a prior (i.e., $\sigma^2 \sim \mathrm{IG}(r, q)$), where the inverse gamma density is

$$P(\sigma^2|r, q) = \frac{1}{r^q\Gamma(q)}(\sigma^2)^{-(q+1)}\exp\left(-\frac{1}{r\sigma^2}\right), \quad (13)$$

where r and q are hyperpriors related to the mean and variance of an inverse gamma distribution such that $r > 2$, $q > 2$, and

$$E[\sigma^2] = \frac{1}{r(q-1)}, \quad (14)$$

$$\mathrm{Var}[\sigma^2] = \frac{1}{r^2(q-1)^2(q-2)}. \quad (15)$$

Given a prior mean μ_{σ^2} and variance $\sigma^2_{\sigma^2}$ for the prior distribution of σ^2 in (11), the corresponding values for r and q can be determined by

$$r = \frac{\sigma^2_{\sigma^2}}{\mu_{\sigma^2}(\mu^2_{\sigma^2} + \sigma^2_{\sigma^2})} \quad (16)$$

$$q = \frac{1}{\mu_{\sigma^2} r} + 1. \quad (17)$$

[26] The prior distribution for τ_c, like that of σ^2, also has physical constraints that need to be captured in the specification. First, τ_c must be positive, and second, if observations of sediment transport are made, then τ_c (a measure of the critical shear stress averaged over the instantaneous Reynolds stresses) is necessarily less than the minimum shear at which transport was observed. Intuitively, the lowest flow at which sediments are moving obviously induces grain shear greater than critical; otherwise, the grains would not move. This is an advantage of the Bayesian approach over nonlinear regression. Constrained optimization of the least squares functions for traditional approaches, though straightforward in principle, is often difficult in practice. The Bayesian approach seamlessly incorporates any constraint within the context of the prior. Given these constraints, a truncated normal (TN) distribution was selected for the prior on τ_c (i.e., $\tau_c \sim \mathrm{TN}(\mu_{\tau_c}, \sigma_{\tau_c})_{\tilde{a}}^{\tilde{b}}$). The density of the truncated normal is

$$P(\tau_c|\mu_{\tau_c}, \sigma_{\tau_c}, \tilde{a}, \tilde{b}) = \frac{\frac{1}{\sigma_{\tau_c}}\phi\left(\frac{\tau_c - \mu_{\tau_c}}{\sigma_{\tau_c}}\right)}{\Phi\left(\frac{\tilde{b} - \mu_{\tau_c}}{\sigma_{\tau_c}}\right) - \Phi\left(\frac{\tilde{a} - \mu_{\tau_c}}{\sigma_{\tau_c}}\right)}, \quad (18)$$

where μ_{τ_c} and σ_{τ_c} are location and shape parameters, $\tilde{a}$ is the lower bound, $\tilde{b}$ is the upper bound, and $\phi()$ and $\Phi()$ are the probability density function and cumulative density function of the standard normal distribution, respectively.

4. Computational Methods

[27] As mentioned, an analytic solution for the posterior shown in (11) is impractical because the truncated normal prior for τ_c will not result in a conjugate (analytically tractable) joint posterior distribution. Therefore, a computational method must be employed that makes it possible to obtain samples from the posterior distribution. In this research, Markov chain Monte Carlo (MCMC) was used to obtain these samples and approximate the posterior distribution of interest.

4.1. Markov Chain Monte Carlo

[28] The purpose of MCMC for model fitting is to construct a Markov chain that has a stationary and ergodic distribution that coincides with the posterior distribution. After a number of iterations, the Markov chain will converge to the target distribution. Once a chain has converged to the target distribution, subsequent realizations from the chain coincide with the target distribution, and thus samples can be collected to approximate posterior quantities. Two different approaches to sampling from the posterior are used: Metropolis-Hastings (M-H) and Gibbs sampling. Because a complete description of these methods is beyond the scope of this paper, only the practical considerations relating to these methods are presented. *Casella and George* [1992] and *Chib and Greenberg* [1995] provide helpful starting points for more detailed descriptions of these algorithms. With reference to the latter condition of convergence, methods for assessing convergence are discussed by *Gelman et al.* [2004] and *Robert* [2007].

4.2. Sampling Algorithm for Transport Model

[29] In order to fit the model presented in (11), an MCMC algorithm using a Gibbs sampler [*Casella and George*, 1992] for the parameter σ^2 and an M-H sampler [*Chib and Greenberg*, 1995; *Hastings*, 1970; *Metropolis et al.*, 1953] for the parameter τ_c were implemented.

[30] Because the inverse gamma prior for σ^2 is conjugate with the normal likelihood, the full conditional distribution for σ^2 can be determined as follows:

$$\begin{aligned}[\sigma^2|\cdot] &\propto \prod_{i=1}^{n}[q_{s,o,i}|\tau_c,\sigma^2][\sigma^2] \\ &\propto \prod_{i=1}^{n} N(q_{s,o,i},\sigma^2)\mathrm{IG}(r,q) \\ &\propto \prod_{i=1}^{n}\left(\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{1}{2}\frac{(q_{s,o,i}-q_{s,i})^2}{\sigma^2}\right\}\right) \\ &\quad\cdot\frac{(\sigma^2)^{-(q+1)}}{r^q\Gamma(q)}\exp\left\{-\frac{1}{r}\frac{1}{\sigma^2}\right\} \\ &\propto (\sigma^2)^{-(n/2+q+1)}\exp\left[-\frac{1}{\sigma^2}\left(\frac{\sum_{i=1}^{n}(q_{s,o,i}-q_{s,i})^2}{2}+\frac{1}{r}\right)\right],\end{aligned} \tag{19}$$

with the bottom proportionality in (19) having the form of an inverse gamma kernel (compare to (13)); therefore, the full conditional distribution for σ^2 in the MCMC algorithm is $[\sigma^2|\cdot] = \mathrm{IG}(\tilde{r},\tilde{q})$. The updated parameters are then

$$\tilde{r} = \left(\frac{\sum_{i=1}^{n}(q_{s,o,i}-q_{s,i})^2}{2}+\frac{1}{r}\right)^{-1} \tag{20}$$

$$\tilde{q} = \frac{n}{2}+q, \tag{21}$$

where r and q are the hyperpriors from (16) and (17).

[31] Because of the inability to analytically determine the full conditional for τ_c, the M-H algorithm was adopted for the critical shear step in the MCMC sampling procedure. This algorithm requires a set of observations $\mathbf{q_{s,o}}$, the parameters for our prior distributions, μ_{σ^2} and $\sigma^2_{\sigma^2}$ (and their corresponding r and q parameterizations from (16) and (17)) for the variance parameter, and μ_{τ_c}, $\sigma^2_{\tau_c}$, $\tilde{a}$, and $\tilde{b}$ for the critical shear parameter. The MCMC algorithm is provided in Appendix A.

5. Model Evaluation

[32] The goals of model evaluation were to determine the ability of the Bayesian transport model (1) to make correct inference on the parameters τ_c and σ^2 (parameter identifiability), (2) to discriminate between competing process models, and (3) to provide reasonable transport predictions. To test this, several strategies were employed. The first was to use synthetic observations whose true underlying parameter values were known. This makes it possible to compare the inferred parameters to the true, known parameters of the synthetic observations. Next, we used transport observations for unisize sediment in a laboratory flume. Unlike the synthetic observations, the underlying parameter values and process model are unknown. Using the laboratory observations, predictive distributions were calculated and compared to the observed transport events. To help discriminate between competing models, we calculated the deviance information criterion (DIC), which is a measure of model fit to parsimony. Last, we evaluated the model's sensitivity to different prior specifications.

5.1. Sensitivity to Prior Specification

[33] The model described in (11) requires the specification of prior distributions for τ_c and σ^2, which are truncated normal and inverse gamma, respectively. The truncated normal distribution is very flexible and can take on numerous different shapes, resulting from the specification of different prior parameter values. The truncated normal distribution is specified as $\tau_c \sim \mathrm{TN}(\mu_{\tau_c},\sigma_{\tau_c})_{\tilde{a}}^{\tilde{b}}$. The hyperpriors, $\mu_{\tau_c}, \sigma_{\tau_c}, \tilde{b}$, and $\tilde{a}$ control the shape of the probability density function. The goal of a sensitivity study is to try different parameterizations of τ_c and assess how they influence posterior inference.

[34] The same argument applies for the prior of σ^2, though the inverse gamma distribution is not nearly as flexible in terms of the types of shapes it can assume. The prior for σ^2 is specified as $\sigma^2 \sim \mathrm{IG}(r,q)$, although, instead of specifying r and q directly, we will specify hyperpriors μ_{σ^2} and $\sigma^2_{\sigma^2}$ for the mean and variance. These hyperpriors are then used to calculate r and q through (16) and (17).

5.2. Posterior Prediction

[35] Checking the posterior predictive distribution (PPD) makes it possible to evaluate how well model predictions

fit the observations. Formally, the posterior predictive distribution is defined as the posterior distribution integrated over the parameters [*Gelman et al.*, 2004]. In our multiparameter sediment transport model, the posterior predictive distribution $[\log(\tilde{\mathbf{q}}_{\mathbf{s,o}})|\log(\mathbf{q}_{\mathbf{s,o}})]$, where $\tilde{\mathbf{q}}_{\mathbf{s,o}}$ is a vector of unobserved events, is found by

$$\iint [\log(\tilde{\mathbf{q}}_{\mathbf{s,o}})|\log(\mathbf{q}_{\mathbf{s,o}}),\tau_c,\sigma^2][\tau_c,\sigma^2|\log(\mathbf{q}_{\mathbf{s,o}})]\mathrm{d}\tau_c\mathrm{d}\sigma^2. \quad (22)$$

The PPD can be used to either make predictions of sediment transport for future observables or to evaluate how well the model fits the actual process observations. The calculation of (22) is done through composition sampling [*Tanner*, 1996]. Using the PPD, we can make predictions under the same conditions as our observations and compare the predictions to the observations.

5.3. Model Discrimination

[36] Because of the large number of prospective transport models, there exists uncertainty as to which model will result in the best fit. The deviance information criterion was described by *Spiegelhalter et al.* [1998] and provides a metric by which model fits and parsimony can be compared. The "best" model is one in which DIC is minimized. The DIC is convenient since it can be easily calculated using MCMC samples and does not require any intractable analytic solutions.

6. Experimental Setup

[37] To achieve the goals specified above, four distinct data scenarios were simulated. These synthetic observations were generated using (6), where realizations for the "true" parameters, τ_c and σ^2 were assigned. Table 1 shows all the parameter values used to simulate observations. Numerous deterministic approaches for estimating τ_c have been published in the literature, with the first by *Shields* [1936] followed by approximations to Shields' data by *Brownlie* [1981], who proposed the following relation for the critical Shields number:

$$\tau_c^* = 0.22Re_p^{-0.6} + 0.06\exp\{-17.77Re_p^{-0.6}\}, \quad (23)$$

where Re_p is the particle Reynolds number,

$$Re_p = \frac{\sqrt{gRD}D}{\nu}, \quad (24)$$

where R is the submerged specific gravity ($R = s - 1$) and ν is the kinematic viscosity of water. The relation in (23) provides a method to specify a plausible τ_c using the previously designated parameter values proposed in Table 1 and is only used for the purpose of data simulation and not for parameter inference.

[38] Prescribing prior parameters for σ^2 is more direct. Values of σ^2 ranging from 0 to approximately 1 provide realistic relationships between shear and sediment discharge. Both low- and high-variance data sets were simulated, with values $\{\sigma^2 : 0.05, 1.10\}$.

[39] In addition to synthetic observations, real flume observations from *Smart* [1984] were used in our Bayesian transport model. The observations reported by *Smart* [1984] are of unisize sediment transport in planar beds and steep slopes and therefore are appropriate for this simple unisize Bayesian sediment transport model.

7. Results and Discussion

7.1. Simulation and Sensitivity Studies

[40] Four distinct data scenarios were simulated for model testing purposes, each consisting of a unique combination of low- and high-variance with many and few observations. Specifically, data scenario 1 consisted of 21 observations with a variance of 0.05 (many low-variance observations); data scenario 2 had six observations with a variance of 0.05 (few low-variance observations); data scenario 3 consisted of 21 observations with a variance of 1.10 (many high-variance observations); last, data scenario 4 had 6 observations with a variance of 1.10 (few high-variance observations). These data scenarios shown in Figure 1.

[41] In addition to four data scenarios, four distinct prior specifications were used in model fits. Histograms of these prior densities are illustrated in Figure 2. These priors were designed to represent a variety of prior scenarios: a diffuse prior (known informally as a noninformative prior) that has a lower bound at zero and an upper bound above the minimum shear at which transport was observed to account for accidental sampling at low flows (prior 1; Figure 2a), a diffuse prior on compact support whose mean is known to be inaccurate (prior 2; Figure 2b), an informative prior with an inaccurate mean and low variance (prior 3; Figure 2c), and an informative prior whose density increases with proximity to the lowest measured shear (prior 4; Figure 2d). Table 2 shows the specified prior means and 95% credible intervals for each τ_c prior scenario. Because each data scenario has a true variance of either 0.05 or 1.10, priors for σ^2 were set at the known value for each data scenario with a small variance ($\mu_{\sigma^2} = \{0.05, 1.10\}$, $\sigma^2_{\sigma^2} = 0.15$) to isolate the effects of prior specification on critical shear. For data scenarios 1 and 2, the prior for σ^2 has a mean of 0.05. For data scenarios 3 and 4 the prior mean is 1.10. Thus, both the high- and low-variance data scenarios will have accounted for the true variance so that the effects of different critical shear priors can be evaluated.

[42] The simulated observations were also used in a more robust test of parameter identifiability. A diffuse prior (prior 1) was assumed for critical shear, and a set of three vague priors (a prior variance of 1000) for σ^2 was tested (see Table 3). Prior 1A has an expected value of 0.001 (an underestimate by a factor of 50 for data scenarios 1 and 2 and an underestimate by a factor of 1100 for data scenarios 3 and 4). Prior 1B has an expected value of 0.275

Table 1. Values for Fixed Parameters

a	b	s	S	T (°C)	ρ (kg m^{-3})	ν (10$_6$ m^2 s^{-1})	g (m s^{-2})	D (mm)	u (m s^{-1})
8	3/2	2.65	0.002	5	1000	1.519	9.81	8.0	$\{u : 0.5\ldots3.0\}$

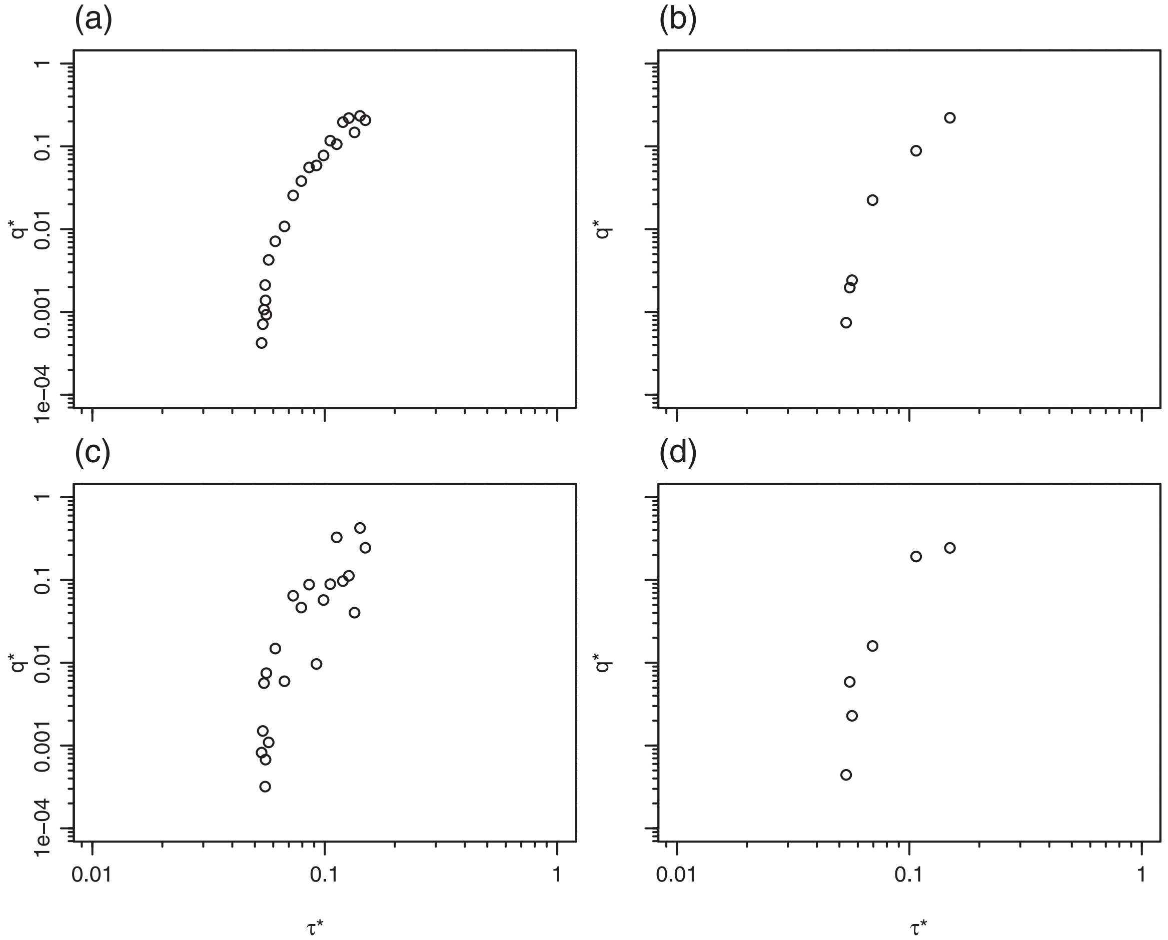

Figure 1. Plot of simulated observation scenarios. Each data scenario is a distinct combination of high and low variance and many and few observations. (a) Data scenario 1: $N = 21$, $\sigma^2 = 0.05$. (b) Data scenario 2: $N = 6$, $\sigma^2 = 0.05$. (c) Data scenario 3: $N = 21$, $\sigma^2 = 1.10$. (d) Data scenario 4: $N = 6$, $\sigma^2 = 1.10$.

(overestimates the true variance of data scenarios 1 and 2 by a factor of 5.5 and underestimates the true variance of data scenarios 3 and 4 by a factor of 4). Prior 1C has an expected value of 5 (100 times larger than the true variance for data scenarios 1 and 2 and 4.5 times larger than the true variance for data scenarios 3 and 4). Table 3 presents the expected values, variances, and 95% credible intervals for these priors.

7.1.1. Inference on Model Parameters

[43] The main purpose of the simulation studies was to determine the robustness of posterior inference on τ_c given different data and prior scenarios. Table 4 contains the simulation results, including the expected value of the posterior estimate of critical shear and critical Shields number along with their 95% credible intervals grouped by scenario. Model misspecification is not a factor in these simulation results since the Meyer-Peter and Müller (MP-M) relation [*Meyer-Peter and Müller*, 1948] was used for both data simulation and parameter estimation.

[44] In each model fit, expert opinion must inform the Bayesian sediment transport model through the prior, and there are many different ways of parameterizing the prior distribution for τ_c. The diffuse prior (prior 1) shown in Figure 2a is the least informative of all the priors; it simply states that critical shear must be greater than zero and is necessarily less than some shear at which sediment transport has been observed. All values within this range are equally probable under the diffuse prior. Prior 2, the diffuse prior on compact support shown in Figure 2b, gives nearly uniform probability to critical shear values over a restricted range of the support in prior 1. The lower bound of prior 2 is no longer zero but was set to 1.55 Pa. The upper bound was also changed from 10 to 4.45 Pa. Given the fact that the true critical shear of the simulated results is 6.72 Pa, the specification of prior 2 is known to exclude the true critical shear value and will prohibit the M-H sampler from exploring the portion of the parameter space outside of this compact support. Prior 3, shown in Figure 2c, was specified to be inaccurate in its mean but not overly limiting in its lower and upper bounds. So, unlike prior 2, the specification of prior 3 permits any value between zero and 10 Pa to be possible, but not with equal probability; the most probable values for τ_c are centered at 3.00 Pa with a small variance. Prior 4, seen in Figure 2d, favors critical shear values proximal to the upper bound of 10 Pa. Prior 4 is useful for situations in which the first observation of sediment transport is supposed to be near critical.

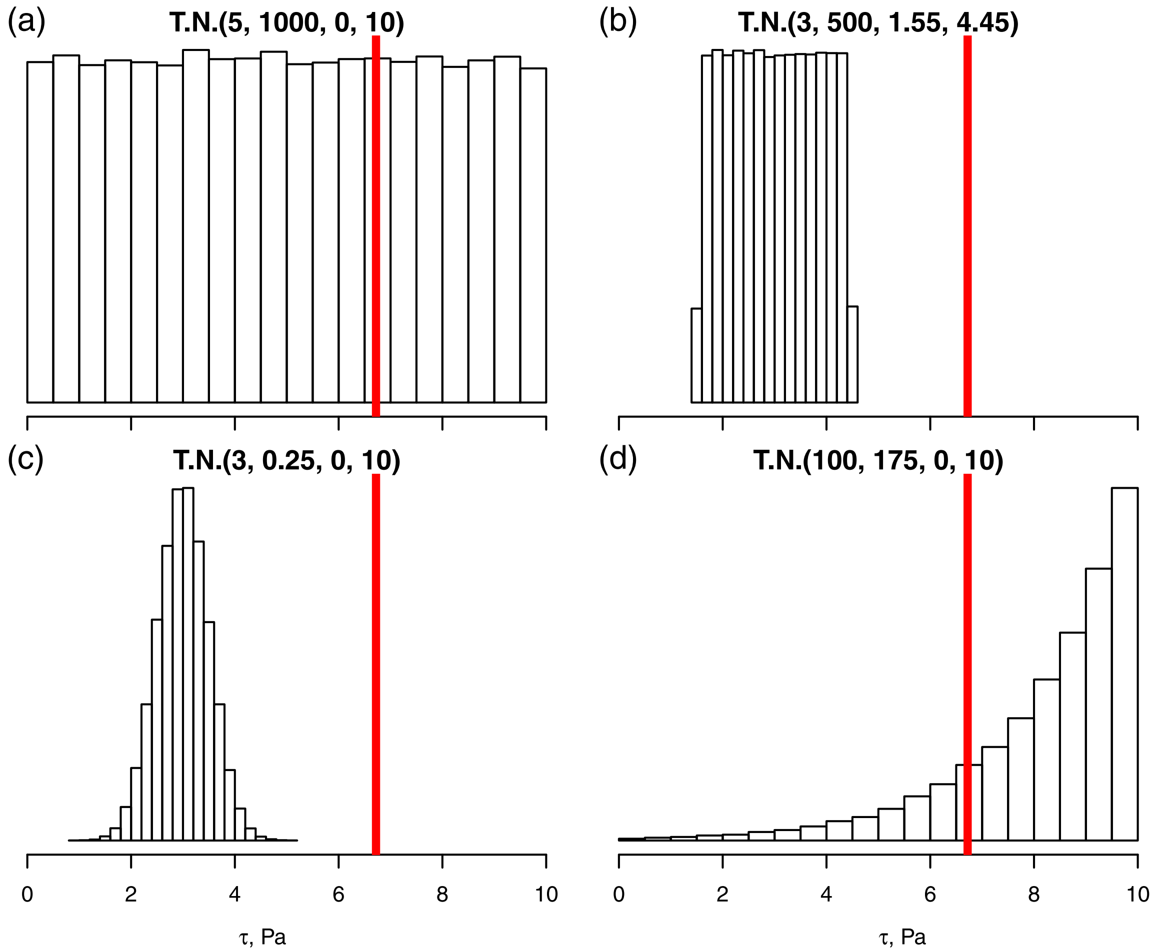

Figure 2. Four prior scenarios. Vertical red line denotes true τ_c in simulated data scenarios. Parameterization follows $\mathrm{TN}(\mu_{\tau_c}, \sigma_{\tau_c}, \tilde{a}, \tilde{b})$. (a) Prior scenario 1, (b) prior scenario 2, (c) prior scenario 3, and (d) prior scenario 4.

[45] Using these prior scenarios, we observe similarities and disparities in the posterior distributions across data scenarios (see Table 4). First, accurate inference was never obtained using the diffuse prior on compact support (prior 2). As was mentioned, the specification of prior 2 restricted the lower and upper bounds, thereby excluding the true critical shear value from the set of possible parameter values. What one observes in Table 4 is that no amount of data can overcome this restriction: data scenario 1 to data scenario 4 all result in unacceptable inference. Unless there are compelling (e.g., physically realistic) reasons to do so, priors using compact support should be avoided. The other priors, however, have varying performance across data scenarios.

[46] Prior 1 can serve as a default for situations where no specific information regarding critical shear is known. The practical effects of prior 1 simply constrain possible values for critical shear between a lower bound of zero and an upper bound of 10 Pa with equal probability. The results in Table 4 show that using a diffuse prior is a very safe modeling strategy to employ in any of the data scenarios. Even for the most limiting data set (data scenario 4), where observations are few and variance is high, a diffuse prior results in acceptable inference on τ_c. The posterior distributions show that where there are more observations, the inferred critical shear values approach the true value used to generate the data.

[47] The prior specification with the widest range of results was prior 3. This prior is informative in that a region of the interval [0, 10] was given higher density than others (see Figure 2c). Prior 3, however, provides misleading information for critical shear. While the true value of τ_c is 6.72 Pa, the 95% credible interval for prior 3 is $[2.02, 3.98]$,

Table 2. Priors for τ_c[a]

Prior	$E[\tau_c]$	95% CI
1	5.00 (0.039)	0.26–9.75 (0.002–0.075)
2	3.00 (0.023)	1.62–4.38 (0.013–0.034)
3	3.00 (0.023)	2.51–3.49 (0.019–0.027)
4	8.17 (0.063)	0.26–9.75 (0.002–0.075)

[a]The critical Shields number is in parentheses. CI, credible interval.

Table 3. Vague Priors for σ^2

Prior	$E[\sigma^2]$	$\mathrm{Var}[\sigma^2]$	95% CI
1A	0.001	1000	1.8×10^{-4} to 0.0042
1B	0.275	1000	0.049–1.132
1C	5.0	1000	0.910–20.40

Table 4. Posterior Inference Results for τ_c[a]

Data Scenario	Prior	Posterior	
		$E[\tau_c]$	95% CI
1	1	6.75 (0.052)	6.70–6.78 (0.052–0.052)
	2	4.18 (0.032)	3.45–4.45 (0.027–0.034)
	3	6.74 (0.052)	6.70–6.78 (0.052–0.052)
	4	6.75 (0.052)	6.70–6.78 (0.052–0.052)
2	1	6.67 (0.052)	6.61–6.71 (0.051–0.052)
	2	3.77 (0.029)	2.01–4.44 (0.016–0.034)
	3	3.32 (0.026)	2.32–4.38 (0.018–0.034)
	4	6.67 (0.052)	6.61–6.71 (0.051–0.052)
3	1	6.63 (0.051)	6.35–6.80 (0.049–0.053)
	2	4.23 (0.033)	3.56–4.44 (0.027–0.034)
	3	4.44 (0.034)	3.01–6.25 (0.023–0.048)
	4	6.64 (0.051)	6.39–6.80 (0.049–0.053)
4	1	6.54 (0.051)	6.04–6.82 (0.047–0.053)
	2	4.02 (0.031)	2.62–4.44 (0.020–0.034)
	3	3.55 (0.027)	2.50–4.63 (0.019–0.036)
	4	6.57 (0.051)	6.05–6.82 (0.047–0.053)

[a]The critical Shields number is in parentheses. For each data scenario the true value of τ_c is 6.72 Pa ($\tau_c^* = 0.052$).

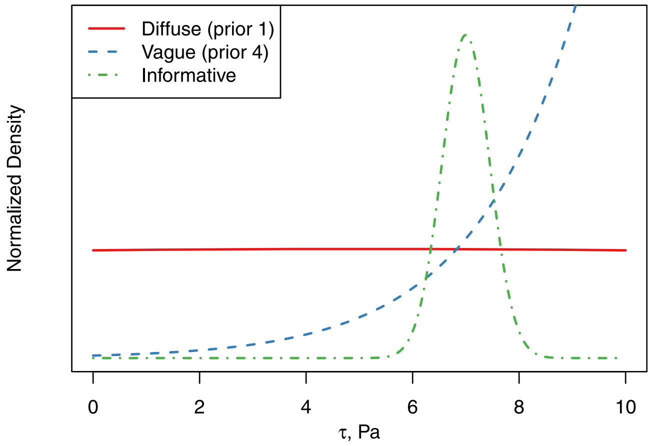

Figure 3. Progressively informative priors for data scenario 4.

as shown in Table 2. Unlike prior 2, which had compact support, the support of prior 3 maintains the lower bound of zero and the upper bound of 10 Pa. While this prior is misleading, it does not prohibit the M-H sampler from exploring the entire interval $[0, 10]$. What we see is that in situations where there are sufficient data (data scenario 1, for example), the misleading prior is overwhelmed by the observations, and the posterior still correctly infers the true value of the critical shear parameter. With fewer observations and increasing variance, however, prior 3 asserts increased influence over the posterior distribution, resulting in unacceptable inference in the remaining data scenarios. Thus, inaccurate priors do not pose insurmountable problems when there are sufficient observations, while situations with few observations will result in unacceptable inference under prior 3.

[48] The final prior (prior 4; Figure 2d) shows uniformly acceptable posterior inference for all data scenarios. Like priors 1, 2, and 3, it has support $[0, 10]$ but gives higher probability to critical shear values close to the upper bound. In data collection scenarios where low transport rates are collected, as suggested by *Wilcock* [2001] for reference shear, prior 4 is particularly useful since expert opinion can inform the model that critical shear is somewhere near the lowest transport observation.

[49] More informative priors, like prior 4, can bolster inference in scenarios with only a few low transport rate observations. Table 5 shows the posterior inference on critical shear using three progressively more informative priors, presented in Figure 3. The diffuse prior is similar to prior 1 in that it gives nearly equal probability to any critical shear value between the lower and upper bounds. The vague prior, similar to prior 4, provides some information to the model regarding where the true value for critical shear is expected. The informative prior gives preference to values close to the true value. What one sees in Table 5 is that using the same observations (data scenario 4), posterior inference is bolstered by prior information. The posterior expectation for critical shear approaches the true value as the information carried in the prior increases. Further, the range of the 95% credible interval also decreases. In sum, well-specified priors provide more precise estimates of critical shear.

[50] The preceding analysis is helpful to isolate the sensitivity of posterior inference on critical shear since the variance parameter was assumed known. In all practical exercises, however (such as the flume experiment whose results follow in section 7.2), this is not reasonable. To this end, we fit three models to data scenarios 1 and 3 using the diffuse prior (prior 1) for critical shear and three vague priors for the variance parameter, each with an inaccurate mean. This analysis makes it possible to determine how sensitive posterior inference on model parameters is affected by inaccurate priors for σ^2 and a diffuse prior for τ_c. Table 6 shows the results of this analysis.

[51] The true variance parameter for data scenario 1 is 0.05, and it is 1.1 for data scenario 3. The priors on σ^2, however, have mean values that range from being 1100

Table 5. Comparison of Priors for Data Scenario 4[a]

Prior	Posterior		
	$E[\tau_c]$	95% CI	CI Range
Diffuse	6.54 (0.051)	6.04–6.82 (0.047–0.053)	0.78 (0.006)
Vague	6.57 (0.051)	6.05–6.82 (0.047–0.053)	0.77 (0.006)
Informative	6.62 (0.051)	6.27–6.84 (0.048–0.053)	0.57 (0.005)

[a]The critical Shields number is in parentheses. For each data scenario the true value of τ_c is 6.72 Pa ($\tau_c^* = 0.052$).

Table 6. Posterior Inference on Model Parameters Using Priors 1A–1C for σ^2 and Prior 1 (Diffuse) for τ_c[a]

Prior	Parameter	Data Scenario 1	Data Scenario 3
1A	τ_c	6.75 (6.71–6.78)	6.64 (6.40–6.80)
	σ^2	0.06 (0.03–0.10)	0.91 (0.50–1.62)
1B	τ_c	6.75 (6.70–6.79)	6.64 (6.40–6.79)
	σ^2	0.08 (0.05–0.15)	0.92 (0.50–1.65)
1C	τ_c	6.73 (6.60–6.83)	6.63 (6.33–6.81)
	σ^2	0.52 (0.91–20.5)	1.36 (0.76–2.42)

[a]Values are the expected value of posterior distribution; values in parentheses denote the 95% credible interval. True critical shear value is 6.72 Pa; true variance for data scenario 1 is 0.05 and is 1.1 for data scenario 3.

times smaller than the true variance to 100 times larger than the true variance. On one extreme, the model fit to data scenario 3 using prior 1A, which assumed a prior mean of 0.001, had a posterior mean of 0.91 and a 95% credible interval that contained the true realized value of 1.1. Prior 1B provided nearly identical inference despite a prior mean of 0.275 compared to 0.001. Prior 1C was an overestimate of σ^2, resulting in a posterior mean closer to the true value of 1.1 but with a relatively wider credible interval. For data scenario 1, whose true variance was 0.05, prior 1A, an underestimate by a factor of 50, performed well, inferring a mean value of 0.06 and a tight credible interval. Prior 1B resulted in posterior inference in which the lower bound of the credible interval contained 0.05, and prior 1C resulted in unacceptable inference since the 95% credible interval did not contain the 0.05. The results in Table 6 indicate that it is far better to have a vague prior for σ^2 that underestimates the true parameter value (even by 1100 times) than it is to overestimate. It should be noted that in all these model fits, regardless of the prior on σ^2, posterior inference on critical shear always contained the true value for τ_c, only with varying degrees of precision, as shown in the associated credible intervals.

7.1.2. Model Discrimination

[52] Because there exist numerous bed load relations in the literature, there is a tyranny of choice when it comes time to model sediment transport. Indeed, *Gomez and Church* [1989] noted that there are more bed load relations than there are good data sets with which to test them. A general criterion for a desirable model is one that is able to provide good fit to observations while minimizing the number of parameters used to obtain that fit. The deviance information criterion provides a metric that quantitatively measures model fit and parsimony, allowing competing models to be compared. A simulation test was performed to verify the utility of the DIC in model discrimination for sediment transport. Synthetic data were generated using the MP-M bed load relation, and two different model fits were performed; one used as a process model the MP-M equation, and the other used the updated parameterization of the MP-M relation proposed by *Wong and Parker* [2006] (W-P), introduced earlier as (7) and (8). These results are presented in Table 7. In comparing models, the magnitude of the DIC is irrelevant; it is used as a relative measure between models. Table 7 shows us that the MP-M correctly infers the true realized value of τ_c, while the W-P model underestimates the realized value. This pattern is repeated in the flume observations for reasons discussed in section 7.2.1. The MP-M model also has a smaller DIC value than the W-P model, a result we would expect because the MP-M relation was used in the data simulation step of this test.

Table 7. Deviance Information Criterion (DIC) Results for Competing Simulation Models[a]

		Posterior	
Model	DIC	$E[\tau_c]$	95% CI
Meyer-Peter and Müller	46.95	6.70 (0.052)	6.63–6.76 (0.051–0.052)
Wong and Parker	72.16	6.25 (0.048)	5.71–6.57 (0.044–0.051)

[a]The critical Shields number is in parentheses. True value of τ_c is 6.72 Pa ($\tau_c^* = 0.052$).

Last, the MP-M model has a credible interval that is narrower than that for the W-P model, so on the basis of these results the MP-M model can be identified as the "better" model. This demonstrates that the DIC can provide a guideline by which competing scientific theories may be compared.

7.2. Flume Observations

[53] The simplicity of the model presented in (6) requires that any flume observations used must meet several criteria. First, the flume observations must be for unisize (or near unisize) sediments; second, because the proposed model does not account for bed forms, the transport must occur over a planar bed; third, since the phenomenon of interest is bed load transport of gravel, a unisize bed composed of grains larger than approximately 8 mm were sought. The experiments of *Smart* [1984] meet these criteria and were used to evaluate the suitability of the proposed model. Specifically, we used the 10.5 mm bed load results with our Bayesian transport model: this experiment consists of 26 sediment transport observations in steep flumes for a nearly uniform distribution of grains.

[54] Unlike the simulation studies, the true parameter values of critical shear and variance are unknown for the *Smart* [1984] data. Given the results from the simulation studies, however, a general modeling strategy was developed. The simulation studies showed that the use of a diffuse prior (prior 1 in Figure 2a) is a defensible default prior. Accordingly, the prior for critical shear was specified as $\tau_c \sim \mathrm{TN}(10, 5000, 0, 20)$. The prior for the variance parameter was specified using judgment as to its mean and variance. A scatterplot of the transport observations demonstrated a low variance, similar to the low-variance scenarios 1 and 2 in the simulation studies. Using these simulation studies as a reference, a prior for the variance was specified to have a mean value of 0.1 and a variance of 1000; the (r, q) hyperpriors were then solved for using equations (16) and (17). These priors are graphed in Figure 4.

[55] Two different models were fit, each using a different process model included as (7) and (8). Tables 8 and 9 summarize the posterior inference on both critical shear and variance for each model fit. Also presented are the DIC values computed for each model fit. Last, Figure 5 shows the posterior predictive distribution for each model fit. Figures 5a and 5c show the 68%, 90%, and 95% credible intervals of the predictive distribution, represented in darkening shades of gray. Figures 5b and 5d show a partitioning of variability in predictions due to (1) the fact that critical shear is being modeled as a random variable instead of a fixed value and (2) the variance parameter σ^2, which is a multiplicative (additive in log space) error term representing measurement error, natural variability of transport, and model misspecification. The combined region in Figures 5b and 5d is the same 95% credible interval shown in Figures 5a and 5c.

7.2.1. Inference on τ_c and σ^2

[56] We see in Table 8 that the inferred critical shear parameter is not independent of the bed load relation used in the model. The MP-M model infers a critical shear of 0.080 Pa, which is reasonable given the steep slopes of the flume studies [*Cao*, 1986]. The W-P model, however, infers an entirely distinct posterior distribution for critical shear; the 95% credible interval for the MP-M models does not

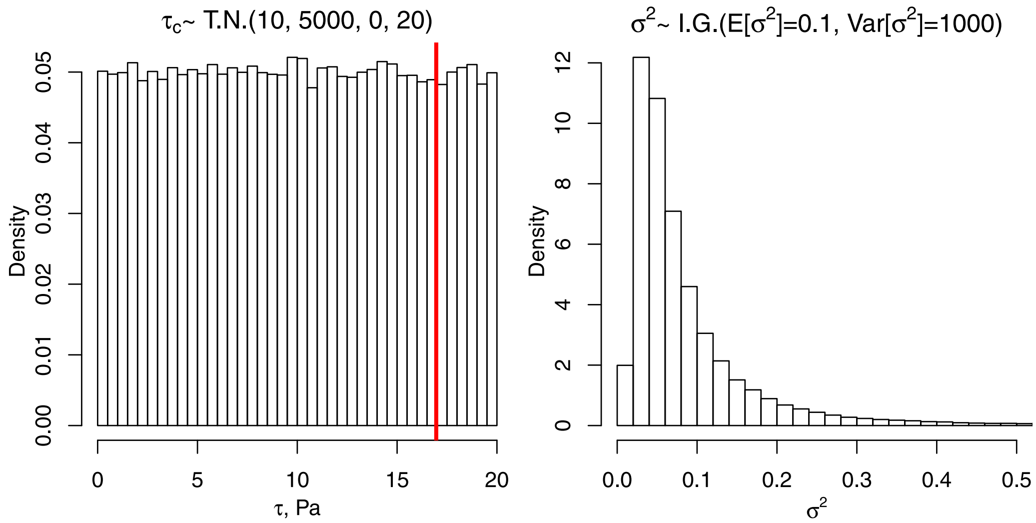

Figure 4. Prior distributions for *Smart*'s [1984] observations. The vertical red line denotes the minimum shear that produced transport.

overlap with that of the W-P model. This is because the MP-M and W-P models have differing values for the *a* and *b* coefficients in an excess shear formulation. At the most basic level, given values of *a, b*, and τ^*, it is expected that solutions for τ_c^* in (1) would yield distinct results. Traditionally, the MP-M and W-P equations assume that the critical Shields number is fixed and known at 0.047. Given the difficulty in specifying the critical Shields number in practice, the present research advocates the measurement of q^* and τ^* and the estimation of τ_c^* based on the measurements and prior knowledge. We see that the existence of a single critical shear value that is portable from one transport relation to another is therefore unreasonable because different process models may result in distinct posterior parameter spaces.

[57] The inferred parameter values for σ^2, as shown in Table 9, however, are in agreement between the MP-M and W-P models, at 0.14. Further, the posterior credible intervals for σ^2 under the MP-M and W-P models are consistent.

7.2.2. Model Discrimination

[58] The DIC values, as reported in Tables 8 and 9, though somewhat inconclusive, seem to indicate that the MP-M model provides the better fit for the number of parameters estimated since it has the smallest value, 25.85 compared to 26.17. The models fitted to the flume observations do not exhibit as large a difference in their DIC values as was the case for the simulated observations in Table 7, possibly inferring only a slight superiority of the MP-M model. Both of the process models used in this research are broadly defined as excess shear models, which have the form $q^* = a(\tau^* - \tau_c^*)^b$ for $\tau_c^* < \tau^*$ and 0 otherwise. The Bayesian formulation presented in this paper models parameters *a*, *b*, and τ^* as fixed and τ_c^* as a random variable and variance σ^2 according to (11). The MP-M and W-P models fit into the definition of the excess shear models since parameters *a* and *b* are specified directly.

7.2.3. Posterior Prediction

[59] A major difference between the predictive results of a purely deterministic approach to sediment transport modeling and those from a Bayesian model is that in the latter, the predictions are functions of random variables, as shown in (22), and are therefore random variables themselves. This characteristic means that predictions in a Bayesian transport model can be represented in terms of probabilities. The plots contained in Figure 5 show the PPDs for the model runs and provide prediction ranges that can be assigned a probability of occurrence. What this shows is that, like purely deterministic approaches, the Bayesian framework can employ deterministic equations to make predictions, but unlike deterministic approaches, it does so using probability distributions for the parameters and predictions. The results in Figure 5 are very informative since the parameter and measurement uncertainty have been fully propagated through the analysis, resulting in a range of credible predictions. The PPD represents what can be reasonably predicted given the parameter, measurement, and structural uncertainty related to the sediment transport phenomenon.

Table 8. Comparison of Process Model Performance on the *Smart* [1984] 10.5 mm Flume Observations for τ_c[a]

Model	DIC	τ_c Posterior	
		$E[\tau_c]$	95% CI
Meyer-Peter and Müller	25.85	13.87 (0.080)	12.52–14.94 (0.072–0.086)
Wong and Parker	26.17	8.00 (0.046)	5.10–10.66 (0.029–0.062)

[a]The critical Shields number is in parentheses.

Table 9. Comparison of Process Model Performance on the *Smart* [1984] 10.5 mm Flume Observations for σ^2

Model	DIC	σ^2 Posterior	
		$E[\sigma^2]$	95% CI
Meyer-Peter and Müller	25.85	0.14	0.08–0.23
Wong and Parker	26.17	0.14	0.08–0.24

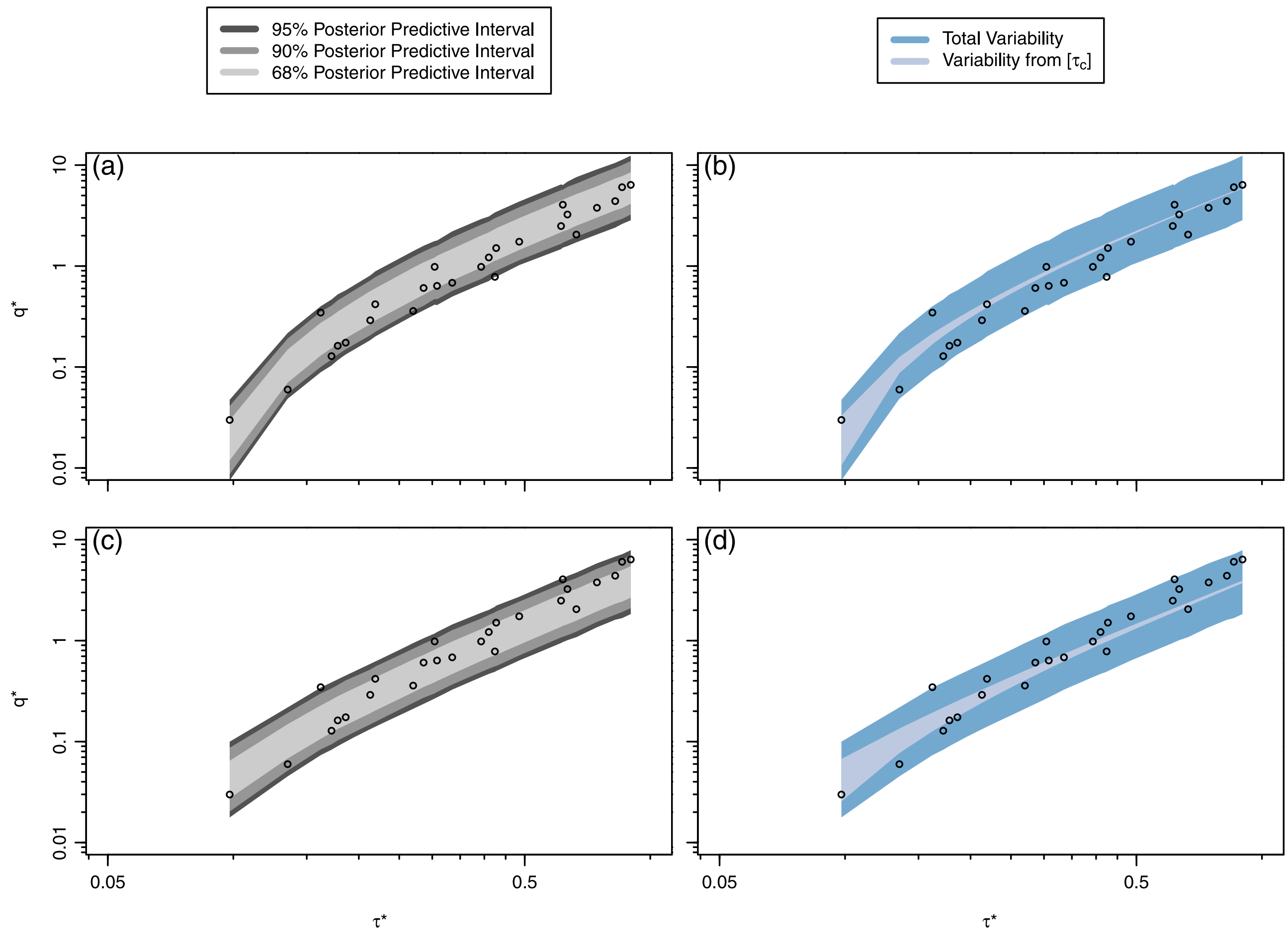

Figure 5. Posterior predictive intervals for *Smart*'s [1984] observations. (a and b) The Meyer-Peter and Müller bed load relation and (c and d) the Wong and Parker relation.

[60] Figures 5a–5d show subtle changes in shape over the range of observed transport events due to the their respective process models. The MP-M model consists of two distinct slopes: higher slope at lower shears that then flattens out at higher shears. The W-P model, however, is generally flatter throughout because τ_c is much lower than for the MP-M equation. The W-P relation tends to underpredict relative to the observed transport rates, whereas the MP-M relation tends to slightly overpredict. This is seen in Figures 5a and 5c since the observed transport rates are located generally on the low side of the PPD. Alternately, the observations in the W-P relation are located generally higher than the PPD. These discrepancies derive from the deterministic models and do not represent an inadequacy of the inferential framework [*Box and Tiao*, 1992].

[61] Figures 5b and 5d show how much variation in the posterior predictions can be partitioned to the underlying distribution of τ_c (as shown in the light band running down the center of the PPD) as well as what can be attributed to the underlying variability, measurement error, and model misspecification shown by the darker area. The Bayesian transport model in (11) is admittedly simple in its construction, and more complicated models may account for a different partitioning of variability to different terms. The implications of this partitioning suggest that some of the variability is reducible (measurement error and model misspecification), while another portion is not (natural variability).

[62] Given the model results using the flume data, the MP-M relation was identified as the most reasonable relation for these observations. This assessment is based in part on the MP-M model having a slightly lower DIC value, but more so because it identifies a tighter credible interval than the W-P relation and further because this credible interval is centered on values that are consistent with our understanding of critical shear in steep flumes.

8. Implications and Conclusions

[63] Having tested our model framework using both simulated and laboratory observations, we now conclude by broadly discussing sediment transport models, characteristics of the Bayesian framework, and the connections between the two in order to identify where a Bayesian approach to sediment transport modeling addresses specific needs of the discipline.

8.1. Implications of Bayesian Model

[64] Early work recognized that a threshold for transport is better described by a statistical distribution than a fixed parameter. In many applied sediment transport problems,

there is considerable difficulty in specifying justifiable values for critical shear, especially when one accounts for the multiplicity of bed arrangements and varying hydraulic conditions that affect incipient motion. To further complicate this, results from this research show that inferred values of critical shear are not independent of the process model used; therefore, a given critical shear value (or distribution of values) that provides the best fit for one relation may result in a very poor fit for a different bed load relation. Additionally, *Wilcock* [2001] notes that when estimating sediment transport rates, strictly formula-based approaches are notoriously inaccurate and sampling campaigns are expensive. One current approach to sediment transport modeling is described by *Wilcock* [2001], who proposes the use of a deterministic transport relation that is calibrated through easily obtained transport samples (i.e., low-transport observations). In this approach, the reference shear parameter, a surrogate for critical shear, is adjusted until a best fit to the observations is obtained. This method employs the use of observations, expert judgment, and a deterministic relationship to provide predictions. Once this model is calibrated, predictions are described by a line fitted to the observations, but deviations from these predictions due to parameter, measurement, and structural uncertainty are not quantified.

[65] A Bayesian sediment transport model is very easily adapted to the approach described above since expert opinion still gets to inform the model as to the values of critical shear through the prior, and transport observations are used to update the expert opinion in a formal way that incorporates variability and error. The reference shear approach requires the specification of a single value for τ_r, whereas a Bayesian approach models critical shear as a random variable arising from a probability distribution. Describing critical shear as a random variable is more consistent with the realities of sediment transport, as discussed by *Shields* [1936], *Einstein* [1950], and, later, *Buffington and Montgomery* [1997]. Further, a Bayesian approach to sediment transport modeling is very efficient since it provides a single framework that robustly (1) estimates model parameters, (2) makes probability-based predictions, (3) incorporates expert knowledge, and (4) provides a means for model discrimination. The results of our model make it possible to explicitly account for predictive uncertainty in sediment transport formulae, given observations, thereby allowing prediction uncertainty to be incorporated in subsequent analyses, such as sediment budgets and river restoration modeling.

[66] Because of its ability to account for uncertainty, this framework would be particularly useful in geomorphological work related to river restoration. *Stewardson and Rutherford* [2008] observed that there exists a tendency for river managers to assume that the physical components of river restoration, such as sediment transport, are well understood with minimal errors. They then demonstrate that there is "unreasonable confidence" [*Stewardson and Rutherford*, 2008, p. 68] in deterministic approaches to sediment transport and conclude that "rarely is any thought given to the best approach to modeling, including the calculation of input parameters to minimize uncertainties in restoration decisions." [*Stewardson and Rutherford*, 2008, p. 75]. Additionally, *Wheaton et al.* [2008] noted that the river management community has largely brushed uncertainties aside. Because of this, the interactions between scientists and river managers are often tenuous [*Wilcock et al.*, 2003]. To ameliorate this, *Wilcock et al.* [2003] concluded that uncertainty in model predictions must be clearly communicated so that management decisions can be better informed. Research in this area can either "ignore the uncertainty and hope that it is not debilitating for the project at hand, or accept the uncertainty and use it as a feature of the research" [*Graf*, 2008]. A Bayesian approach to sediment transport modeling, such as that offered here, robustly addresses the challenges expressed above.

[67] Further, the interpretation of the predictive distributions is intuitive. Whereas traditional statistical methods report confidence intervals that represent the percent of time the constructed interval will contain the true, fixed parameter value, Bayesian credible intervals are statements of probability; for example, the probability that the parameter of interest is contained in this interval is $(1-\alpha)\%$. This simple interpretation of model results facilitates the "clear communication of uncertainty" recommended by *Wilcock et al.* [2003].

[68] Last, because Bayesian models provide posterior predictive distributions, intuitive assessments of predictive uncertainty are avoided. Of the numerous studies published on decision making under uncertainty, *Tversky and Kahneman* [1973] discuss how intuitive assessments of confidence are biased by representativeness and input consistency. A conclusion of their research is that individuals are prone to experience a high degree of confidence in highly fallible judgments; stated otherwise, intuitive assessments of the credibility of predictions are usually highly overconfident. The posterior predictive distribution developed in (22), however, avoids the shortcomings of intuitive assessments of uncertainty and provides a robust measure of the uncertainty in model predictions.

8.2. Model Extensions for Future Implementations

[69] The simple model disclosed in this research demonstrates how a Bayesian sediment transport model may be developed. The present model is simple for pedagogical purposes and certainly does not represent the full complexity that is possible. Some specific extensions that are being pursued by the current authors are multifraction bed load transport, which incorporates description of how different grain sizes are transported; inclusion of turbulence structures and hiding effects; mixture models and model averaging for model inference or to accommodate unknown error structures [*Xiao et al.*, 2011]; development of hierarchical implementations that account for different error sources; evaluation of implications in constructing sediment mass balances; and incorporating uncertainty into model forcing observations (such as streamflow).

[70] The established hydrology literature demonstrates potential uses for the Bayesian framework in sediment transport as well. *Renard et al.* [2010] provide a comprehensive example of how Bayesian approaches can be leveraged to inform model selection and error structure and how uncertainty can inform a model; such an approach applied to sediment transport would be a valuable contribution. Pre-MCMC hydrology literature illustrates the potential use of decision theory in hydrology and how it can inform decision on the basis of customizable loss functions that account for the capital and maintenance costs of overdesign and the monetary consequences of inadequate design

[*Davis et al.*, 1972; *Duckstein and Szidarovszky*, 1977], thereby providing an optimal decision from a decision-theoretic perspective. Given the broad challenges faced in the sediment transport community, it is our belief that Bayesian models can provide a tool for innovation.

Appendix A: MCMC Algorithm

[71] 1. Select a starting value for critical shear, $\tau_c(0)$.

[72] 2. For iterations, $k = 1, 2, \ldots N$, (1) calculate $q_s|\tau_c(k-1)$ using (6), (2) calculate updated parameters $\tilde{r}$ and $\tilde{q}$ using (20) and (21), (3) obtain kth sample for $\sigma^2 \sim \text{IG}(\tilde{r}, \tilde{q})$, (4) sample $\tau_{c^*} \sim \text{TN}[\tau_c(k-1), \sigma^2_{\text{tune}}]_{\tilde{a}}^{\tilde{b}}$, (5) calculate $q_{s^*}|\tau_{c^*}$ using (6), (6) calculate the ratio ρ via

$$\rho = \frac{[\mathbf{q_{s^*}}|\mathbf{q_o}, \sigma^2(k)][\tau_{c^*}|\mu_{\tau_c}, \sigma^2_{\tau_c}]}{[\mathbf{q_s}|\mathbf{q_o}, \sigma^2(k)][\tau_c(k-1)|\mu_{\tau_c}, \sigma^2_{\tau_c}]},$$

and (7) accept or reject proposed value τ_{c^*}:

$$\tau_c(k) = \begin{cases} \tau_{c^*}, & \text{with probability } \rho, \\ \tau_c(k-1), & \text{with probability } 1-\rho. \end{cases}$$

[73] This algorithm is repeated N times until an adequate number of samples is obtained from the target distribution. The total number of samples from the target distribution is equal to the number of samples N less the number of iterations required for burn-in $N_{\text{burn-in}}$(the number of samples required for convergence on target density). These samples from the target distribution can then be used to make inference on the parameters of interest: τ_c and σ^2.

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